Solutions to Math 232 Final, Version 1 (Blue/Grey)

Spring 2012, Simon Fraser University

1(a).

$$A + A^T = \begin{pmatrix} -5 & 2 & 1 \\ -4 & 1 & 3 \\ 1 & 0 & 3 \end{pmatrix} + \begin{pmatrix} -5 & -4 & 1 \\ 2 & 1 & 0 \\ 1 & 3 & 3 \end{pmatrix} = \begin{pmatrix} -10 & -2 & 2 \\ -2 & 2 & 3 \\ 2 & 3 & 6 \end{pmatrix}$$

1(b). $\det(B) = (1)(2)(1) = 2$ (the product of the diagonal entries) since B is upper triangular. So $\det(B^3) = \det(BBB) = \det(B)\det(B)\det(B) = \det(B)^3 = 2^3 = 8$.

1(c).

$$\begin{aligned} e^{5\pi(1+i)/2} &= e^{5\pi/2}e^{5\pi i/2} \\ &= e^{5\pi/2}\Big(\cos(5\pi/2) + i\sin(5\pi/2)\Big) \\ &= e^{5\pi/2}(0+i) \\ &= 0 + e^{5\pi/2}i \end{aligned}$$

2(a). To find a basis of the row space, we can row reduce

$$\begin{pmatrix} 0 & 1 & -3 \\ 2 & 3 & 1 \\ 1 & 2 & -1 \end{pmatrix}$$

We switch the first row with the third to get

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ 0 & 1 & -3 \end{pmatrix}$$

and then subtract two times the first row from the second to get

$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 0 & 1 & -3 \end{pmatrix}$$

and then add the second row to the third to get

$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

and add twice the second row to the first to get

$$\begin{pmatrix} 1 & 0 & 5 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

and scale the second row by -1 to get

$$\begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$

Now we are in reduced row echelon form, so a basis for the row space is $\{(1, 0, 5), (0, 1, -3)\}$.

2(b). The rank of A is the dimension of the row space, which is the number of vectors in a basis of the row space. By part (a), this is 2.

2(c). Since $\text{rank}(A) + \text{nullity}(A)$ equals the number of columns of A, and $\text{rank}(A) = 2$ from part (b), the nullity of A is 1.

3(a). $[T] = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 0 & -1 & -3 \end{pmatrix}.$

3(b). $$T(-1, 2, 1) = [T]\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 0 & -1 & -3 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -5 \end{pmatrix}$$

3(c). The kernel of T is the same as the null space of $[T]$, i.e., we solve

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 0 & -1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

by row reducing

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & -1 & -3 & 0 \end{array}\right)$$

So we subtract two times the first row from the second to get

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -1 & -3 & 0 \\ 0 & -1 & -3 & 0 \end{array}\right)$$

and subtract the second row from the third to get

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

and add the second row to the first to get

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & -1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

and scale the second row by -1 to get

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

so x and y are pivot variables and z is free. Set $t = z$ and then $x = z = t$ and $y = -3z = -3t$. The kernel is just this general solution: the set of all vectors $t\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$ with $t \in \mathbb{R}$. So $\{(1, -3, 1)\}$ is a basis for the kernel of T.

3(d). This is not a one-to-one transformation, since the kernel is not $\{\mathbf{0}\}$.

3(e). T is a linear operator, and linear operators are either both one-to-one and onto, or neither. Since T is not one-to-one, it is also not onto.

4(a). The counterclockwise rotation by $\pi/3$ is

$$\begin{pmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & \cos(\pi/3) \end{pmatrix} = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}.$$

The reflection across the y-axis is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. So, remembering the correct order to write the matrices (first transformation rightmost), we want

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}.$$

4(b). $\det \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} = \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = -\frac{1}{4} - \frac{3}{4} = -1$. This is a reflection since rotations have determinant $+1$ and reflections have determinant -1.

5. B_1 is not a basis because the vector $(1, 1, 0)$ is not in the plane: $1 + 1 + 2(0) = 2 \neq 0$. B_3 is not a basis since it is not linearly independent: the second vector is 2 times the first vector. B_2 is a basis because its vectors are in the plane: $1 + (-1) + 2(0) = 0$ and $2 + 2 + 2(-2) = 0$, they are linearly independent (they are not scalar multiples of each other), and their total count (two vectors) equal the dimension of the plane (2).

6(a). The rank is the dimension of the row space (or of the column space), which could be any subspace of $\mathbb{R}^3$. So rank could be 0, 1, 2, or 3, but nothing else is possible.

6(b). The rank is the dimension of the row space, so we can row reduce

$$\begin{pmatrix} 1 & 1 & t \\ 1 & t & 1 \\ t & 1 & 1 \end{pmatrix}$$

Then subtract the first row from the second

$$\begin{pmatrix} 1 & 1 & t \\ 0 & t-1 & 1-t \\ t & 1 & 1 \end{pmatrix}$$

and subtract t times the first row from the third

$$\begin{pmatrix} 1 & 1 & t \\ 0 & t-1 & 1-t \\ 0 & 1-t & 1-t^2 \end{pmatrix}$$

and add the second row to the third

$$\begin{pmatrix} 1 & 1 & t \\ 0 & t-1 & 1-t \\ 0 & 0 & 2-t-t^2 \end{pmatrix}$$

and if we factor the quadratic in the lower right corner, we have

$$\begin{pmatrix} 1 & 1 & t \\ 0 & t-1 & 1-t \\ 0 & 0 & -(t-1)(t+2) \end{pmatrix}$$

So if $t = 1$, we have

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and the row space is spanned by $\{(1, 1, 0)\}$, and it 1-dimensional. If $t = -2$, then we have

$$\begin{pmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

and the row space is spanned by $\{(1, 1, -2), (0, -3, 3)\}$, which is a linearly independent set (since the two vectors in it are not scalar multiples of each other). So the row space is 2-dimensional when $t = -2$. If t is neither 1 or -2, then

$$\begin{pmatrix} 1 & 1 & t \\ 0 & t-1 & 1-t \\ 0 & 0 & -(t-1)(t+2) \end{pmatrix}$$

has all diagonal entries nonzero, and so its three rows are linearly independent (we could then continue row reducing to eventually get the identity matrix). So the dimension of the row space is 3 when $t \neq 1, -2$.

In summary the rank is 1 when $t = 1$, the rank is 2 when $t = -2$, and the rank is 3 for all other values of t.

7(a). Since W is a plane in $\mathbb{R}^3$, the set of vectors orthogonal to it is a line. Or, formally, $\dim(W) = 2$, and since $\dim(W) + \dim(W^\perp) = \dim(\mathbb{R}^3) = 3$, we see that $\dim(W^\perp) = 1$, so $W^\perp$ is a line.

7(b). One can read off from the equation of the plane that its normal vector is $(1, 2, -1)$. So $\{(1, 2, -1)\}$ is a basis of $W^\perp$.

7(c). The easiest way to do this is to project $(1, 4, 0)$ onto $W^\perp$ first. Since $W^\perp$ is the span of $\{(1, 2, -1)\}$,

$$\begin{aligned} \text{proj}_{W^\perp}(1, 4, 0) &= \frac{(1, 4, 0) \cdot (1, 2, -1)}{||(1, 2, -1)||^2}(1, 2, -1) \\ &= \frac{9}{6}(1, 2, -1) \\ &= (3/2, 3, -3/2) \end{aligned}$$

and then

$$\begin{aligned}\mathrm{proj}_W(1,4,0) &= (1,4,0) - \mathrm{proj}_{W^\perp}(1,4,0)\\ &= (1,4,0) - (3/2,3,-3/2)\\ &= (-1/2,1,3/2)\end{aligned}$$

7(d). The point is $(-1/2,1,3/2)$. This is because it is the projection onto W of $(1,4,0)$.

7(e). The distance between $(1,4,0)$ and $\mathrm{proj}_W(1,4,0)$ is just the length of their difference, that is the length of $\mathrm{proj}_{W^\perp}(1,4,0) = ||(3/2,3,-3/2)|| = \sqrt{(3/2)^2+3^2+(-3/2)^2} = \sqrt{27/2}$.

8(a). Since $(1,1,0)\cdot(3,0,-1) = 3+0+0 = 3 \neq 0$, the two vectors are not orthogonal, so they do not form an orthogonal basis.

8(b). By the Gram-Schmidt procedure, we can form an orthogonal basis $\{\mathbf{v}_1,\mathbf{v}_2\}$ by setting $\mathbf{v}_1 = (1,1,0)$ and

$$\begin{aligned}\mathbf{v}_2 &= (3,0,-1) - \frac{(3,0,-1)\cdot(1,1,0)}{||(1,1,0)||^2}(1,1,0)\\ &= (3,0,-1) - \frac{3}{2}(1,1,0)\\ &= (3,0,-1) - (3/2,3/2,0)\\ &= (3/2,-3/2,-1)\end{aligned}$$

So our orthogonal basis is $\{(1,1,0),(3/2,-3/2,-1)\}$.

8(c).— We normalize the vectors from our orthogonal basis $\mathbf{v}_1,\mathbf{v}_2$ in the previous step to get $\frac{1}{||\mathbf{v}_1||}\mathbf{v}_1 = \frac{1}{\sqrt{2}}(1,1,0)$ and $\frac{1}{||\mathbf{v}_2||}\mathbf{v}_2 = \sqrt{\frac{2}{11}}(3/2,-3/2,-1)$. So our orthonormal basis is $\left\{\frac{1}{\sqrt{2}}(1,1,0), \sqrt{\frac{2}{11}}(3/2,-3/2,-1)\right\}$.

9(a). We compute

$$A\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}7 & 8\\-4 & -5\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix} = -1\begin{pmatrix}-1\\1\end{pmatrix},$$

so that $(1,-1)$ is indeed an eigenvector of A with eigenvalue -1. We compute

$$A\begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}7 & 8\\-4 & -5\end{pmatrix}\begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}6\\-3\end{pmatrix} = 3\begin{pmatrix}2\\-1\end{pmatrix},$$

so that $(2, -1)$ is indeed an eigenvector of A with eigenvalue 3.

9(b). The matrix P should have the eigenvectors as its columns and the matrix Λ should have the eigenvalues on the diagonal, with the eigenvectors of P and the eigenvalues of Λ listed in corresponding order. Thus we can use

$$P = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$$

and

$$\Lambda = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$$

or we could reverse the order of columns in P, but then we would also need to reverse the order of the diagonal entries of Λ.

9(c). Since $A = P\Lambda P^{-1}$, we have $A^{57} = (P\Lambda P^{-1})^{57} = P\Lambda^{57}P^{-1}$, so the matrix we are looking for is

$$D = \Lambda^{57} = \begin{pmatrix} (-1)^{57} & 0 \\ 0 & 3^{57} \end{pmatrix}$$

10(a). The characteristic polynomial of A is

$$\det(\lambda I - A) = \det \begin{pmatrix} \lambda - 2 & 1 & 0 \\ -1 & \lambda & 0 \\ 0 & 0 & \lambda - 2 \end{pmatrix}$$

and we use cofactor expansion on the last row to obtain

$$\begin{aligned} \det(\lambda I - A) &= (\lambda - 2) \det \begin{pmatrix} \lambda - 2 & 1 \\ -1 & \lambda \end{pmatrix} \\ &= (\lambda - 2)(\lambda^2 - 2\lambda + 1) \\ &= (\lambda - 2)(\lambda - 1)^2 \end{aligned}$$

so the eigenvalues are 2 and 1.

10(b). Looking at the number of times each root appears in the characteristic polynomial, we see that algebraic multiplicity of eigenvalue $\lambda = 2$ is 1 and the algebraic multiplicity of $\lambda = 1$ is 2.

10(c). The eigenspace for $\lambda = 2$ is $(2I - A)$, so we solve the system

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

by row reducing

$$\left(\begin{array}{rrr|r} 0 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We exchange the first and second rows

$$\left(\begin{array}{rrr|r} -1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

then scale the first row by -1

$$\left(\begin{array}{lrr|r} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

and then add two times the second row to the first

$$\left(\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

so x and y are pivot variables, and z is free. Set $t = z$ and then $x = 0$ and $y = 0$. So the general solution is

$$t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

so the eigenspace for $\lambda = 2$ is the span of a single nonzero vector, and so is 1-dimensional. So the geometric multiplicity for $\lambda = 2$ is 1.

The eigenspace for $\lambda = 1$ is $(I - A)$, so we solve the system

$$\begin{pmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

by row reducing

$$\left(\begin{array}{rrr|r} -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right)$$

We scale the first row by -1

$$\left(\begin{array}{rrr|r} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right)$$

and add the first row to the second to get

$$\left(\begin{array}{rrr|r} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right)$$

then exchange the second and third rows to get

$$\left(\begin{array}{rrr|r} 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

and scale the second row by -1 **to get**

$$\left(\begin{array}{rrr|r} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

so x **and** z **are pivot variables, and** y **is free. Set** $t = y$ **and then** $x = y = t$ **and** $z = 0$**. So the general solution is**

$$t\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

so the eigenspace for $\lambda = 1$ **is the span of a single nonzero vector, and so is** 1**-dimensional. So the geometric multiplicity for** $\lambda = 1$ **is** 1**.**

10(d). A **is not diagonalizable because we have only two independent eigenvectors since the geometric multiplicities of** $\lambda = 2$ **and** $\lambda = 1$ **are** 1 **and** 1**. Or, one could say that the sum of the geometric multiplicities is** $1+1 = 2$**, which is less than the size** 3 **of our square matrix. Or, one could note that the geometric multiplicity**

of $\lambda = 1$ **is less than the algebraic multiplicity of** $\lambda = 1$**.**

11(a). The degree of the polynomial is $4 + 2 + 1 + 3 = 10$**, so the matrix must have** 10 **rows and** 10 **columns. (Only square matrices have characteristic polynomials).**

11(b). The trace is the sum of the eigenvalues (repeated according to algebraic multiplicity), so it is $4(0) + 2(1) + 1(3) + 3(-5) = 0 + 2 + 3 - 15 = -10$**.**

11(c). A **is not invertible since** 0 **is one of its eigenvalues (so** $\det(A)$ **must be** 0**).**

11(d). Yes. Since A **has** 1 **as one of its eigenvalues, the associated eigenvectors (which are not $\mathbf{0}$) are fixed points.**